

A different approach to anisotropic spherical collapse with shear and heat radiation

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Abstract

In order to study the type of collapse, mentioned in the title, we introduce a physically meaningful object, called the horizon function. It directly enters the expressions for many of the stellar characteristics. The main junction equation, which governs the collapse, transforms into a Riccati equation with simple coefficients for the horizon function. We integrate this equation in the geodesic case. The same is done in the general case when one or another of the coefficients vanish. It is shown how to build classes of star models in this formulation of the problem and simple solutions are given.

1 Introduction

Gravitational collapse is one of the main topics in relativistic astrophysics. The first example was given by Oppenheimer and Snyder [1] who studied the collapse of a dust cloud. It has energy density but no pressures. Then followed studies of the collapse of perfect or anisotropic fluid spheres [2]. Such models have clearly defined boundary, where the interior solution matches the exterior Schwarzschild solution. The main junction condition is the vanishing of the radial pressure at the surface of the star. The process of collapse, however, is highly dissipative, required to account for the enormous binding energy of the resulting object [3]. Thus a more realistic scenario is the collapse with heat flow [4] or pure radiation [5]. Spherical collapse is described in the general case by a diagonal metric with three independent components, g_{tt} , g_{rr} and $g_{\theta\theta}$. The exterior solution is the Vaidya shining star [6]. In the diffusion approximation (heat flow) the main junction condition changes. At the boundary surface the radial pressure should equal the heat flux. In the streaming approximation (pure radiation) it remains the same - vanishing of the radial pressure. The main junction condition is a non-linear differential equation in partial derivatives (along radius

and time) for the three metric components and follows from the matching of the second fundamental forms. It is in fact a constraint that reduces the number of independent metric components to two. It has the same form both in the heat flow case and in the null radiation case. The other junction conditions give expressions for different characteristics of the exterior solution in terms of the interior one.

For simplicity, shearless fluid is discussed quite often, because then the differential equation involves only the two components g_{tt} and g_{rr} . When the fluid is perfect they have to satisfy also another differential equation, called the isotropy condition. Even in this case the amount of interior solutions is enormous [7]. Global solutions for perfect fluids were given soon after the new junction equation was derived [8], [9]. Separation of variables allows to integrate the equation and see explicitly the evolution of the star from a static model to a black hole. A number of such solutions were found later, e.g. [10]. It has been noticed that there is an additional, simple solution, linear in time for g_{rr} , which leads to eternal collapse and a horizon never appears. This happens both for perfect [5], [7], [11] and imperfect fluids [12]. An example of this no-horizon phenomenon is given for the geodesic case ($g_{tt} = 1$) without shear, too [13]. Another approach to the junction equation uses its Lie point symmetries and provides different classes of solutions. This procedure was applied to perfect fluid models [14] and to models with anisotropic pressures [15].

When shear is present, the general three component metric should be used, which further complicates the differential equation. The geodesic case, however, is simpler, involving again two components, g_{rr} and $g_{\theta\theta}$. The first exact geodesic solution with radiation was obtained by [16]. After that it was noticed [17] that the junction condition is a Riccati equation for g_{rr} . It is of first order and is not integrable in general. When some of its coefficients are set to zero, it reduces to integrable equations and simple expressions for g_{rr} may be found. In this approach one finds first the metric, which is fixed, because comoving coordinates are used, and then obtains the physical characteristics of the model from the Einstein equations. Two simple regular solutions in separated variables were derived. The previous solution is regained when certain parameters are set to zero. Later, more general exact solutions, depending on arbitrary functions of the coordinate radius, were given [18]. They encompass the previous solutions. In a recent development the realm of analytic solutions was further expanded by studying the Lie point symmetries of the junction condition [19]. This results in generalized traveling waves and self-similar solutions. Finally, all solutions of the equation were found with the help of two generating functions, based on the approach discussed in the present paper [20]. All previous solutions were obtained by choosing the generating functions in a proper way.

The general case with shear has been attacked both numerically and analytically. Chan and co-workers used the initial method of separated variables. In the presence of shear the time evolution equation is not integrable and has to be solved numerically [21], [22], [23], [24], [25]. In some cases the initial static model evolves into a black hole, but in others the star burns out completely, radiating its mass away, so that no horizon ever forms and the final result is flat

spacetime.

The analytic approach is based on the fact that g_{rr} satisfies a Riccati equation, like in the geodesic case. Its reduction to a linear, Bernoulli, or simpler Riccati equation is exploited to find solutions in elementary functions [26], [27], [28].

Another class of shearing, radiative, anisotropic fluid models are the so-called Euclidean stars [29]. Their thermal behaviour was studied [30]. A class of Euclidean stars with no horizon was found [31]. Finally, the Lie symmetry approach has been applied to the general case with shear and classes of new solutions were produced. They were called generalized Euclidean stars since some of the usual Euclidean stars are obtained as a subcase [32].

L. Herrera and co-authors have studied different aspects of relativistic stars with shear, like expansionfree collapse [33], the evolution equation of the shear and the extension of the mass formalism to the radiative case [34], the cause of energy-density inhomogeneity [35] and the collapse in the post-quasistatic approximation [36].

The main idea of the approach used in the present paper is to transform the junction equation into an equation for a physically meaningful object, which we call the horizon function. It is directly related to the redshift and the formation of a horizon, which means the appearance of a black hole as the end product of collapse. It enters the expression for the mass of the star, the heat flow and the luminosity at infinity.

In Sect. 2 we present the Einstein equations, which in the anisotropic case are expressions for the energy density, the radial and the tangential pressure and the heat flow. The definitions of the shear, the expansion, the horizon function and the redshift are given. The relation between the mass of the star and the horizon function is clarified. The main results of the matching to the exterior Vaidya solution are shown. The most important of them is a differential equation involving the metric components. We show that it is the same in the diffusion and in the streaming approximations. The luminosities at the star's surface and at infinity are written, as well as the surface temperature. We also use another version of the Einstein equations, known as the mass formalism, to give additional expressions for the heat flow and the energy density, which are connected to the mass. In Sect. 3 the junction equation is written as a Riccati equation for the horizon function with simple coefficients and the process of building a star model solution is clarified. In Sect. 4 all geodesic solutions are derived from a single generating function. In Sect. 5 the general Riccati equation is reduced to a special one, without a linear term, which is integrable. The simplest solution is demonstrated. In Sect. 6 the general Riccati equation is reduced to a linear one, which is integrated too. A solution, for a linear in time star radius is studied. Sect. 7 represents a discussion.

2 Stellar characteristics

The collapse of an anisotropic fluid sphere with shear is described by the following metric

$$ds^2 = -A^2 dt^2 + B^2 dr^2 + R^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (1)$$

where A , B and R are independent functions of time t and the radius r only. The spherical coordinates are numbered as $x^0 = t$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \varphi$. The energy-momentum tensor, describing dissipation through heat flow and null fluid, reads

$$T_{ik} = (\mu + p_t) u_i u_k + p_t g_{ik} + (p_r - p_t) \chi_i \chi_k + q_i u_k + u_i q_k + \varepsilon l_i l_k. \quad (2)$$

Here μ is the energy density, p_r is the radial pressure, p_t is the tangential pressure, u^i is the four-velocity of the fluid, χ^i is a unit spacelike vector along the radial direction, q^i is the heat flow vector, also in the radial direction, ε is the energy density of the null fluid and the vector l^i is null. In comoving coordinates we have

$$u^i = \delta_0^i, \quad \chi^i = B^{-1} \delta_1^i, \quad q^i = q \chi^i, \quad l^i = u^i + \chi^i. \quad (3)$$

The Einstein field equations read

$$\mu + \varepsilon = \frac{1}{A^2} \left(\frac{2\dot{B}}{B} + \frac{\dot{R}}{R} \right) \frac{\dot{R}}{R} - \frac{1}{B^2} \left(\frac{2R''}{R} + \frac{R'^2}{R^2} - \frac{2B'R'}{BR} - \frac{B^2}{R^2} \right), \quad (4)$$

$$p_r + \varepsilon = -\frac{1}{A^2} \left[\frac{2\ddot{R}}{R} - \left(\frac{2\dot{A}}{A} - \frac{\dot{R}}{R} \right) \frac{\dot{R}}{R} \right] + \frac{1}{B^2} \left(\frac{2A'}{A} + \frac{R'}{R} \right) \frac{R'}{R} - \frac{1}{R^2}, \quad (5)$$

$$\begin{aligned} p_t = & -\frac{1}{A^2} \left[\frac{\ddot{B}}{B} + \frac{\ddot{R}}{R} - \frac{\dot{A}}{A} \left(\frac{\dot{B}}{B} + \frac{\dot{R}}{R} \right) + \frac{\dot{B}\dot{R}}{BR} \right] + \\ & + \frac{1}{B^2} \left[\frac{A''}{A} + \frac{R''}{R} - \frac{A'B'}{AB} + \left(\frac{A'}{A} - \frac{B'}{B} \right) \frac{R'}{R} \right], \end{aligned} \quad (6)$$

$$qB + \varepsilon = -\frac{2}{AB} \left(\frac{\dot{B}R'}{BR} - \frac{\dot{R}'}{R} + \frac{\dot{R}A'}{RA} \right), \quad (7)$$

Here the dot means a time derivative, while the prime stands for a radial derivative.

For the line element (1) the shear σ , the four-acceleration a_1 and the expansion scalar Θ are given by

$$\sigma = \frac{1}{3} \left(\frac{\dot{R}}{R} - \frac{\dot{B}}{B} \right), \quad (8)$$

$$a_1 = \frac{A'}{A}, \quad \Theta = \frac{2\dot{R}}{R} + \frac{\dot{B}}{B}. \quad (9)$$

Next, we introduce the important object H , which we call "the horizon function" for reasons to become clear later:

$$H = \frac{R'}{B} + \frac{\dot{R}}{A}. \quad (10)$$

The mass m , entrapped within radius r is given by the expression [37]

$$m = \frac{R}{2} \left[1 + \left(\frac{\dot{R}}{A} \right)^2 - \left(\frac{R'}{B} \right)^2 \right]. \quad (11)$$

On the stellar surface Σ it becomes the mass of the star. The compactness parameter reads $u = m/R$. Eq (11) can be rewritten using H

$$\frac{2m}{R} = 1 - H^2 + \frac{2\dot{R}}{A}H. \quad (12)$$

The exterior spacetime is given by the Vaidya shining star solution

$$ds^2 = - \left[1 - \frac{2M(v)}{\rho} \right] dv^2 - 2dv d\rho + \rho^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (13)$$

where $M(v)$ is the mass of the star measured at time v by an observer at infinity, while ρ is the outer radius. Both solutions should be joined smoothly at Σ , which leads to the following junction conditions [38], [39]:

$$R_\Sigma = \rho_\Sigma(v), \quad (14)$$

$$m_\Sigma = M_\Sigma, \quad (15)$$

$$(p_r)_\Sigma = (qB)_\Sigma. \quad (16)$$

Eq (16) should be satisfied by A , B and R , while the other equations are definitions of different stellar characteristics. When there is only null radiation, q vanishes and the radial pressure should vanish at the surface. This condition was used in many works with null fluid radiation [5]. However, when $\varepsilon \neq 0$ Eqs.(5,7) show that in terms of the metric Eq (16) is restored. Thus it is universal for heat and null radiation. Therefore, in the following we set $\varepsilon = 0$ and study only the heat radiation case. Now when q vanishes we get collapse without radiation and the exterior solution is the Schwarzschild vacuum solution.

Some important stellar characteristics are defined on the surface of the star. These are the redshift z_Σ

$$z_\Sigma = \frac{1}{H_\Sigma} - 1, \quad (17)$$

the surface luminosity Λ_Σ and the luminosity at infinity Λ_∞

$$\Lambda_\Sigma = \left(\frac{1}{2} q B R^2 \right)_\Sigma, \quad (18)$$

$$\Lambda_\infty = H_\Sigma^2 \Lambda_\Sigma. \quad (19)$$

The temperature at the surface is given by

$$T_\Sigma^4 = \frac{(qB)_\Sigma}{8\pi\delta}, \quad (20)$$

where δ is some constant.

Additional formulas are obtained from the mass formalism [40]. In its original form it gives expressions for the derivatives of the mass function in the non-radiative case. It has been expanded to the dissipative case [34]

$$\frac{\dot{m}}{A} = -\frac{1}{2} \left(p_r \frac{\dot{R}}{A} + qB \frac{R'}{B} \right) R^2, \quad (21)$$

$$\frac{m'}{R'} = \frac{1}{2} \left(\mu + qB \frac{B\dot{R}}{AR'} \right) R^2. \quad (22)$$

On the star's surface Eq (21) becomes, using the junction condition (16),

$$\left(\dot{m} = -\frac{1}{2} qB H A R^2 \right)_\Sigma. \quad (23)$$

Collapse takes place with heat radiated towards outside, hence, $q > 0$. Then m decreases since energy is lost. Eq (22) gives another formula for the energy density throughout the star

$$\mu = \frac{2m'}{R^2 R'} - \frac{qB^2 \dot{R} R^2}{AR'}. \quad (24)$$

It is seen that the star properties have simpler expressions when written in terms of H . The redshift is positive during collapse. Then Eq (17) shows that $0 \leq H_\Sigma \leq 1$. When $H_\Sigma = 0$ we obtain from Eq (12) and the junction conditions

$$\left(1 - \frac{2m}{R} \right)_\Sigma = \left(1 - \frac{2M(\nu)}{\rho} \right)_\Sigma = 0. \quad (25)$$

This signals the appearance of a horizon and a black hole within it, which is the typical end of gravitational collapse. This explains why we call H the horizon function. The redshift becomes infinite, while the luminosity at infinity drops to zero. The point in time when collapse starts is taken as t_i (usually it is $-\infty$). There H_Σ should have some positive value $H_{\Sigma i}$, less or equal to 1. Thus during the collapse to a black hole the horizon function decreases to zero and $\dot{H}_\Sigma \leq 0$.

3 Simplification of the junction equation

With the help of Eqs (5,7) the main junction Eq (16) becomes on the star surface

$$\begin{aligned} -\frac{2}{AB} \left(\frac{\dot{B}R'}{Br} - \frac{\dot{R}'}{R} + \frac{\dot{R}A'}{RA} \right) &= -\frac{1}{A^2} \left[\frac{2\ddot{R}}{R} - \left(\frac{2\dot{A}}{A} - \frac{\dot{R}}{R} \right) \frac{\dot{R}}{R} \right] + \\ &+ \frac{1}{B^2} \left(\frac{2A'}{A} + \frac{R'}{R} \right) \frac{R'}{R} - \frac{1}{R^2}. \end{aligned} \quad (26)$$

It is not hard to see that this is a Riccati equation for B . Regrouping the terms we get

$$\dot{B} = \left(\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} - \frac{2\dot{A}\dot{R}}{AR} + \frac{A^2}{R^2} \right) \frac{R}{2AR'} B^2 + \left(\frac{\dot{R}'}{R'} - \frac{A'\dot{R}}{AR'} \right) B - \frac{A}{2} \left(\frac{R'}{R} + \frac{2A'}{A} \right). \quad (27)$$

This is exactly Eq(11) from Ref. [26]. There is no general solution of the Riccati equation, although it is integrable in many concrete cases. Furthermore, when the coefficient in front of B^2 vanishes it becomes a linear equation, which is solvable for any given A and R . When the free term vanishes it becomes a Bernoulli equation, which is also solvable. Finally, when the coefficient before the linear term vanishes, we get another Riccati equation, which sometimes is simpler. Examples are given in Refs. [26], [27], [28].

We shall go along a different way. When B satisfies a Riccati equation the same is true for $\frac{a}{B} + b$, where a and b are arbitrary functions of t and r . Let us take first $P \equiv R'/B$. Eq (27) becomes a Riccati one for P :

$$\dot{P} = \left(\frac{A}{2R} + \frac{A'}{R'} \right) P^2 + \frac{A'\dot{R}}{AR'} P - \frac{R}{2A} \left(\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} - \frac{2\dot{A}\dot{R}}{AR} + \frac{A^2}{R^2} \right). \quad (28)$$

Next we can write

$$P = H - \frac{\dot{R}}{A}. \quad (29)$$

Replacing this expression in the previous equation yields a Riccati equation for the horizon function H on the star surface. It seems to be more complicated than Eq (28). However, the opposite is true. Many of the terms containing time derivatives of R cancel each other. The final result is rather simple:

$$\dot{H} = \left(\frac{A}{2R} + \frac{A'}{R'} \right) H^2 - \left(\frac{A}{R} + \frac{A'}{R'} \right) \frac{\dot{R}}{A} H - \frac{A}{2R}. \quad (30)$$

Only one term with \dot{R} survives. While A is related to the four-acceleration and R is the physical radius of the star as seen from the outside, B is just a metric component without direct physical meaning. On the contrary, as we have explained, H has a lot of physical applications and it's important to have a simple equation for it. Eq (17) shows that the redshift also satisfies a Riccati equation. However, it is more complicated and inconvenient to work with than Eq (30).

Once again we can obtain solutions by choosing $A(r, t)$ and $R(r, t)$ explicitly, taking their derivatives and setting $r = r_\Sigma$ to obtain the coefficients of Eq (30) in explicit form. We can also reduce it to a linear equation or a special Riccati equation without a linear term. The Bernoulli equation does not appear here because $A \neq 0$.

Multiplying Eq (30) by R one obtains a similar equation for $D \equiv RH$

$$\dot{D} = \left(\frac{A}{2R} + \frac{A'}{R'} \right) \frac{D^2}{R} - \frac{A'\dot{R}}{AR'} D - \frac{A}{2}. \quad (31)$$

The expression for $B(t, r)$ follows from the definition of H , Eq (10)

$$B = \frac{R'}{H - \dot{R}/A}. \quad (32)$$

During the collapse the radius of the star shrinks, hence, $\dot{R} < 0$. Since B is positive, we need $R' > 0$. This is exactly the condition for the absence of shell crossing singularities [41]. Using Eq (30) we can find another expression for the mass of the star. Let us multiply Eq (12) by $A/R + A'/R'$, provided the latter combination does not vanish. Utilising Eq (30) as an expression for \dot{R} , we get

$$\left(\frac{A}{R} + \frac{A'}{R'}\right) \frac{2m}{R} = \frac{A'}{R'} (1 + H^2) - 2\dot{H}, \quad (33)$$

trading \dot{R} for \dot{H} . Sometimes this formula is useful.

In the formalism described above we find collapse solutions along the following chain. First we take positive functions $A(t, r)$ and $R(t, r)$, satisfying $\dot{R} < 0$ and $R' > 0$. Then a positive solution for H is found from Eq (30) which must be a decreasing function, starting from some H_i . This allows to obtain immediately the redshift from Eq (17). Next we find the mass m from Eq (12) and B from Eq (32). Now it is possible to measure the heat flow q from Eq (23) and the energy density μ . Then the two luminosities and the temperature on the star surface are given by Eqs (18,19,20). The two pressures are found from the Einstein equations (5,6). Some of these quantities are defined on the surface, some throughout the star. For realistic solutions we must have $A, B, R, R', m, m', q, H, \mu, p_r > 0$ and $\dot{R}, \dot{H}, \dot{m} < 0$. The energy conditions should hold, too.

4 All geodesic solutions

The power of Eq (30) for the horizon function is best revealed in the case of geodesic solutions. The particles of the fluid move along geodesics and the four-acceleration a_1 is zero. Eq (9) shows that A depends only on time. It can be set to 1 by a time transformation of the line element. Then all terms with A' disappear. Eq (30) becomes

$$2R\dot{H} = H^2 - 2\dot{R}H - 1, \quad (34)$$

Eq (34) has already been found [20] from the equation for the Z function [18]. Eq (31) transforms into

$$\dot{D} = \frac{1}{2R^2} D^2 - \frac{1}{2}. \quad (35)$$

This is a Riccati equation for D , but an algebraic one for R and D may be taken as the generating function of the solutions. Then

$$R = \frac{D}{\sqrt{2\dot{D} + 1}}, \quad (36)$$

Now, since this equation holds on the star's surface, we can promote the constants in it into functions of r and add to the r.h.s. the term $G(t, r) = g(r) F(t, r)$ where g and F are arbitrary, as long as R is positive. In addition $g(r_\Sigma) = 0$ and the term G does not show up on the surface. This is a second generating function allowed by Eq (36). We set it to zero for simplicity. Then Eq (36) and the definition of D yield

$$H = \sqrt{2\dot{D} + 1}. \quad (37)$$

Eq (32) with $A = 1$ gives an expression for B . Eq (33) becomes

$$m = -R^2 \dot{H}. \quad (38)$$

These relations have been found in a different way too [20]. Thus a zero of H signals the formation of a black hole, while a zero in its time derivative signals the burning away of the mass due to radiation and the appearance of flat spacetime - another end of the collapse. The decrease in the mass is given by Eq (23) with $A = 1$. One can give expressions for all stellar characteristics in terms of D and its time derivatives.

This, however, is not the only generation function. One can integrate Eq (37) and obtain an analogue of Eq (36)

$$R = \frac{1}{2H} \left(\int H^2 dt - t \right). \quad (39)$$

Now the horizon function plays also the role of a generating function. Further, we can replace H with the redshift from Eq (17) to obtain

$$R = 2(1+z) \left(\int \frac{dt}{(1+z)^2} - t \right). \quad (40)$$

Thus the role of a third generating function is played by the redshift, which is a physical observable.

An important issue is to restore from this general formalism the particular solutions, found in the past [20]. The generalized traveling waves and self-similar solutions [19] may be found by choosing D to depend on an special intermediate function of t and r . The linear in time solution of Ref. [18] seems unphysical, because it cannot have both the mass and the heat flow positive. The analogy with the shearless case seems to break here. The other solution given in this reference is physical, depends on several functions of r and can evolve either to a black hole or to flat spacetime, depending on how we choose these functions.

5 Special Riccati equation

Let A and R satisfy on the star's surface Σ the equation

$$\frac{A}{R} + \frac{A'}{R'} = 0. \quad (41)$$

This eliminates the linear term in Eq (30) and it becomes

$$\dot{H} = \left(\frac{A}{2R} + \frac{A'}{R'} \right) H^2 - \frac{A}{2R}. \quad (42)$$

Eq (41) provides a relation between A and R :

$$R = \frac{K(t)}{A}, \quad (43)$$

where $K(t)$ is an arbitrary positive function. It makes Eq (42) integrable

$$\frac{dH}{1+H^2} = -\frac{A^2}{2K} dt, \quad (44)$$

which yields the solution

$$H = tg \frac{1}{2} \int_t^{t_f} \frac{A^2}{K} dt. \quad (45)$$

Here t_f is the end time of collapse where $H = 0$ and a black hole, covered by a horizon is formed. Eq (44) gives

$$\frac{2R}{A} = -\frac{1+H^2}{\dot{H}}. \quad (46)$$

We may take A and H as independent arbitrary functions and find R from this equation. Let $A = A(r)$. Then R is a solution with separated variables and the above equation yields upon differentiation

$$\frac{2\dot{R}}{A} = -2H + (1+H^2) \frac{\ddot{H}}{H^2}. \quad (47)$$

Replacing this expression in Eq (12) for the mass we obtain

$$\frac{2m}{R} = 1 - 3H^2 + (1+H^2) \frac{H\ddot{H}}{\dot{H}^2}. \quad (48)$$

Thus the compactness parameter $u = m/R$ depends only on H and its time derivatives. Now, since on the star's surface $A(r_\Sigma)$ and $A'(r_\Sigma)$ are some constants, H_Σ becomes the only generating function for this class of solutions. One must be careful to choose it in such a way that all above mentioned inequalities are satisfied in order to obtain physically realistic solutions. We shall demonstrate that this is not so easy.

Let us choose the simplest case when $\ddot{H}_\Sigma = 0$. Then

$$H_\Sigma = c_1 - c_2 t, \quad (49)$$

where c_i are positive constants and $c_1 > c_2$. The collapse starts at some initial time t_i and ends at $t_f = c_1/c_2$. H_Σ is positive and decreasing, \dot{H}_Σ is negative. Hence R_Σ in Eq (46) is positive and can be written as

$$R_\Sigma = \frac{A_\Sigma}{2c_2} (1 + H_\Sigma^2). \quad (50)$$

It decreases with time, therefore $\dot{R}_\Sigma < 0$. Eqs (48,50) give for the mass

$$m_\Sigma = \frac{A_\Sigma}{4c_2} (1 - 2H_\Sigma^2 - 3H_\Sigma^4). \quad (51)$$

This is positive as long as $H_\Sigma < 1/\sqrt{3}$. This may be arranged by choosing c_i appropriately for any t_i . However, the time derivative of the mass is

$$\dot{m}_\Sigma = A_\Sigma H_\Sigma (1 + 3H_\Sigma^2), \quad (52)$$

which is always positive. Thus $q_\Sigma < 0$ and energy is pumped into the star from the outside until it turns into a black hole when $2u_\Sigma = 1$. This does not seem realistic. The study of concrete solutions will be continued elsewhere.

6 Linear equation

The quadratic term in Eq (30) disappears and it becomes a linear equation when

$$\frac{A}{2R} + \frac{A'}{R'} = 0. \quad (53)$$

It leads to the relation

$$R = \frac{K(t)}{A^2}, \quad (54)$$

where, as before, $K(t)$ is an arbitrary positive function. This case was discussed in Ref. [26] and for B it leads to a Bernoulli equation with very complicated solution. Here Eq (30) simplifies considerably

$$2R\dot{H} + \dot{R}H + A = 0 \quad (55)$$

and is linear both in H and in R . Let us suppose for simplicity that collapse starts at $t = 0$ when a non-trivial heat flow q appears, due to processes taking place in the star, being static before that time. The solution of the general linear differential equation

$$g\dot{H} = f_1 H + f_0 \quad (56)$$

can be written in the form

$$H = e^F \left(H_i + \int_0^t e^{-F} \frac{f_0}{g} dt \right), \quad (57)$$

where

$$F = \int_0^t \frac{f_1}{g} dt, \quad (58)$$

clearly showing that the initial value of the horizon function is H_i . It differs slightly from the formula usually given in handbooks [42]. For Eq (55) we get

$$F = -\frac{1}{2} \ln \frac{R}{R_i}, \quad R_i = R(t=0), \quad (59)$$

$$H = \sqrt{\frac{R_i}{R}} \left(H_i - \frac{1}{2\sqrt{R_i}} \int_0^t \frac{A^2}{\sqrt{K}} dt \right). \quad (60)$$

Provided A and K are given, we can find R, H, B, m, q and the other characteristics of the star.

In the shearless case a simple linear in time solution for $R = rB$ leads to "eternal collapse" without horizon [7], [11], [12]. It is natural to study similar solution in the case with shear, that we are discussing. Thus, let us take

$$R_\Sigma = R_i - bt \quad (61)$$

on the star's surface, with positive constants R_i, b . Let $A = A(r)$. Then the integration in Eq (60) yields

$$H_\Sigma = \frac{A_\Sigma}{b} - \sqrt{\frac{R_i}{R_\Sigma}} \left(\frac{A_\Sigma}{b} - H_i \right) = c - \sqrt{\frac{R_i}{R_\Sigma}} \varepsilon, \quad (62)$$

where $A_\Sigma/b \equiv c$, and $c - H_i \equiv \varepsilon$. Here c and ε are positive constants. Obviously $\dot{R}_\Sigma < 0$ and $\dot{H}_\Sigma < 0$. At some t_f we get $H_{f\Sigma} = 0$ for the corresponding $R_{f\Sigma}$, that is, a black hole appears:

$$R_{f\Sigma} = \frac{\varepsilon^2}{c^2} R_i \quad (63)$$

Eq (12) gives for the mass

$$2m_\Sigma = -R_i \varepsilon^2 + 2\sqrt{R_i R_\Sigma} \varepsilon \frac{1+c^2}{c} - (1+c^2) R_\Sigma \quad (64)$$

and its time derivative

$$2\dot{m}_\Sigma = (1+c^2) \dot{R}_\Sigma \left(\frac{\varepsilon}{c} \sqrt{\frac{R_i}{R_\Sigma}} - 1 \right). \quad (65)$$

The second bracket is negative because of Eq (63) and $R_\Sigma > R_{f\Sigma}$. Hence, the mass of the star is a monotonously increasing function, like in the previous section.

Now we can prove that at least m_Σ can be chosen positive. First, at $R_{f\Sigma}$ we get

$$2m_{f\Sigma} = \frac{\varepsilon^2 R_i}{c^2}, \quad (66)$$

which is positive. Second, at the beginning of collapse we have $R_\Sigma = R_i$ and

$$2m_{\Sigma i} = R_i \left[2\varepsilon \frac{1+c^2}{c} - \varepsilon^2 - (1+c^2) \right]. \quad (67)$$

This expression is positive when

$$H_i^2 + \frac{2}{c} H_i - 1 < 0. \quad (68)$$

One possible solution of this inequality is

$$H_i = \frac{c}{4}, \quad c < 2\sqrt{2}. \quad (69)$$

We conclude that the simplest solution in this section is unphysical because $\dot{m}_\Sigma > 0$.

7 Discussion

The main purpose of this paper was to reformulate the main junction equation in terms of the physically important function H . This is possible because, originally, this condition represents a Riccati equation for the metric component B . It is known that under a fractional linear transformation of the unknown function the Riccati equation for it preserves its nature, but the coefficients attached to the different terms change [42]. Since H is such a transformation of B , it also satisfies a Riccati equation. One expects this to be useless, because the coefficients seem to become much more complicated. Due to reasons, unknown to us, there are miraculous cancellations and most of the terms cancel each other. The resulting equation is simpler and directly guides the behaviour of H and its zero, where a horizon and a black hole appear.

Eq (17) gives a simple relation between the observable redshift and the horizon function H , which holds in any case. It represents a fractional linear transformation, this time of H . Hence, the redshift also satisfies a Riccati equation. More generally, every statement about H can be translated into a statement about the redshift. Eq (19) shows that the square of H transforms the surface luminosity (depending mainly on the heat flow) into the luminosity at infinity and serves as a measure of its faintness. The horizon function also serves as a main ingredient of the mass formula and the compactness parameter. The time derivative of the mass gives a simpler expression for the heat flow, which is fundamental for the surface temperature and luminosity of the star.

In this different approach we again start with analytic expressions for A and the luminous radius of the star R and then solve analytically the equation for H . A signal for its great potential is that for geodesic fluid motion all solutions may be found explicitly with the help of two generating functions. One of them can be chosen to be $D \equiv RH$, H or the redshift z [20]. Thus, observing the time evolution of the redshift, e.g., is enough to restore the evolution of the star characteristics on the surface.

In the general case, setting certain coefficient to zero changes the Riccati equation into an integrable equation. This requires a simple relation between A and R . In some cases H plays alone the role of a generating function, like in the geodesic case, (see Eq (46) with $A = A(r)$). The simplest solutions, unfortunately, are astrophysically unrealistic (the heat flow is negative), but still help to see how a black hole appears. We relegate the presentation of realistic solutions to future work.

This different approach works when A, B and R are independent apriori. In the shearless case $R = rB$ and the junction equation is not Riccati, but a

more complex one. Usually the absence of shear is accepted in order to simplify the problem. We see that the opposite is true. The general case with shear is simpler than the shearless one.

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